A combinatorial model for highest weights of finite dimensional representations of $\mathfrak{gl}(m, n)$

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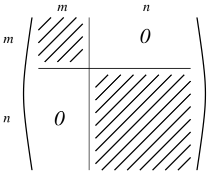
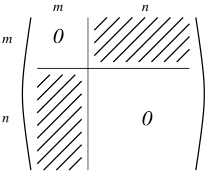
- $\mathfrak{g}_{\bar{0}} =$  ; $\mathfrak{g}_{\bar{1}} =$ 

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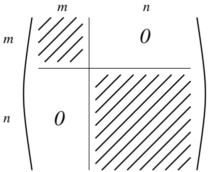
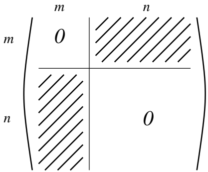
- $\mathfrak{g}_{\bar{0}} =$ 

; $\mathfrak{g}_{\bar{1}} =$ 

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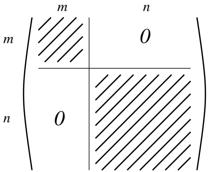
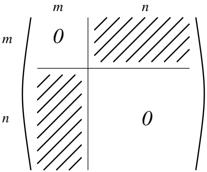
- $\mathfrak{g}_{\bar{0}} =$ 

; $\mathfrak{g}_{\bar{1}} =$ 

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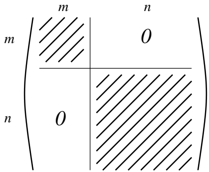
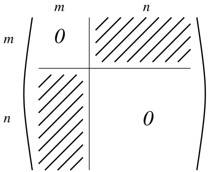
- $\mathfrak{g}_{\bar{0}} =$ 

; $\mathfrak{g}_{\bar{1}} =$ 

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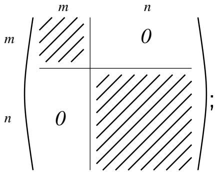
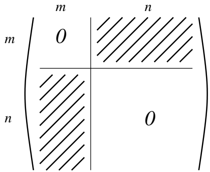
- $\mathfrak{g}_{\bar{0}} =$ 

; $\mathfrak{g}_{\bar{1}} =$ 

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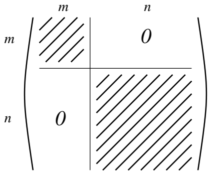
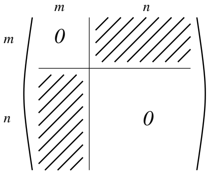
- $\mathfrak{g}_{\overline{0}} =$ 

; $\mathfrak{g}_{\overline{1}} =$ 

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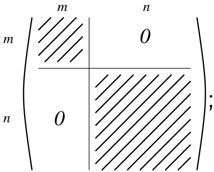
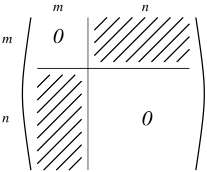
- $\mathfrak{g}_{\bar{0}} =$ 

; $\mathfrak{g}_{\bar{1}} =$ 

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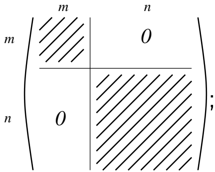
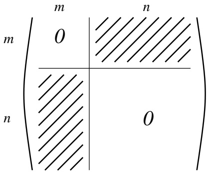
- $\mathfrak{g}_{\bar{0}} =$ 

; $\mathfrak{g}_{\bar{1}} =$ 

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- Roots: $\{\varepsilon_i - \varepsilon_j\}$, $\{\delta_i - \delta_j\}$; $\{\pm(\varepsilon_i - \delta_j)\}$,
- Choice of simple roots $\Leftrightarrow$ ordering of $\{\varepsilon_1, \ldots, \varepsilon_m, \delta_1, \ldots, \delta_n\}$.

Combinatorial model

Arc diagram:

Combinatorial model

Arc diagram:

$\bullet \quad \times \quad \times \quad \bullet \quad \times \quad \bullet \quad \times \quad \bullet \quad \times$

Sequence of m $\bullet$'s and n $\times$'s encodes the choice of base (simple roots).

Combinatorial model

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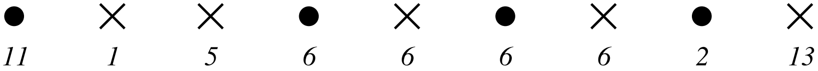

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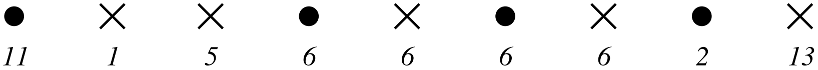

Integers (call them *entries*) under the nodes give coordinates of the weight

$$\lambda + \rho = 11\varepsilon_1 + 6\varepsilon_2 + 6\varepsilon_3 + 2\varepsilon_4 - \delta_1 - 5\delta_2 - 6\delta_3 - 6\delta_4 - 13\delta_5$$

Combinatorial model

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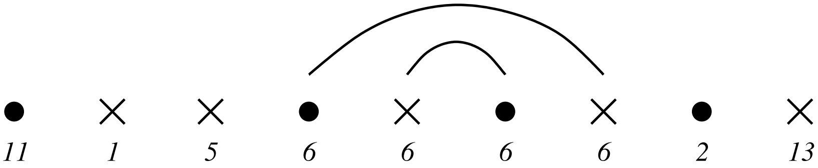

Arcs denote a choice of "$(\lambda + \rho)$-maximal isotropic subset" :

$$S = \{\varepsilon_2 - \delta_4, \delta_3 - \varepsilon_3\}.$$

Combinatorial model

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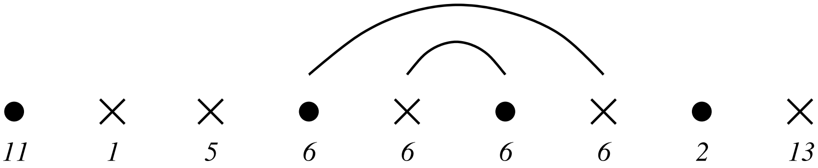

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Combinatorial model

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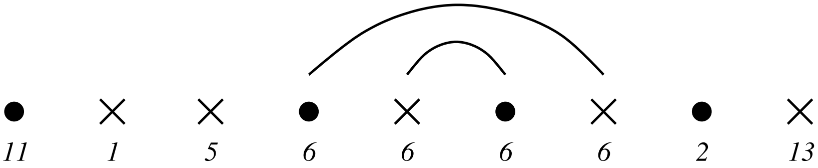

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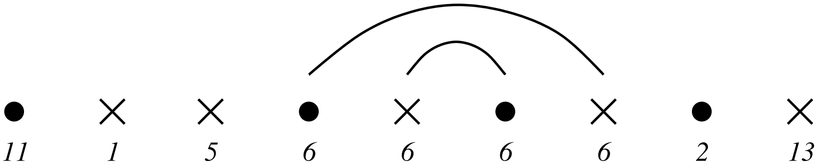

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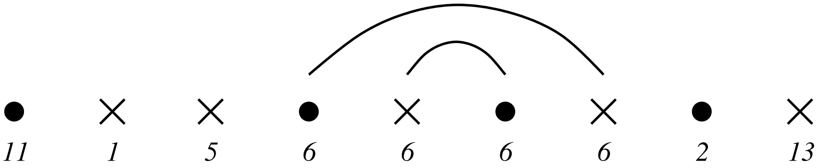

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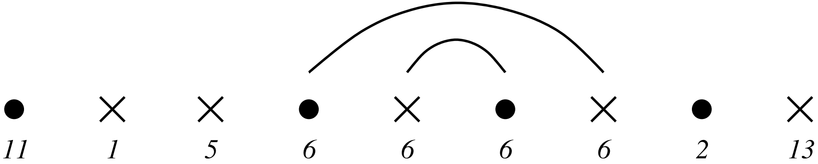

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- the entries corresponding to the ends are equal,
- S is maximal with respect to the above properties.

Combinatorial model

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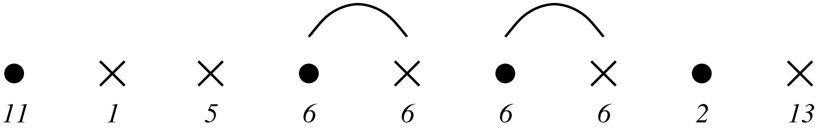
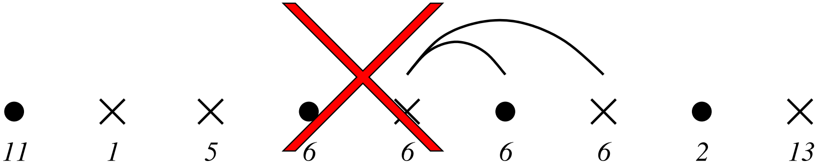


Combinatorial model

Not an arc diagram:



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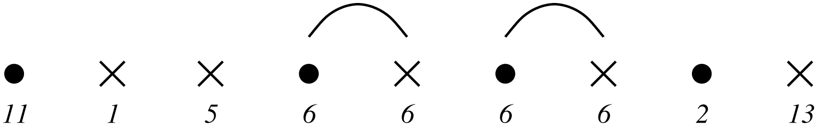
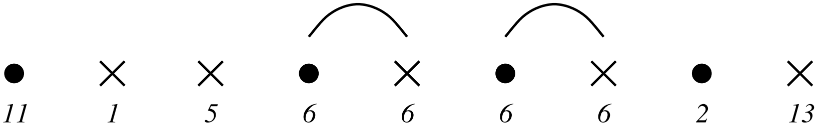


Moves:

Combinatorial model

Arc diagram:



Moves:

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Combinatorial model

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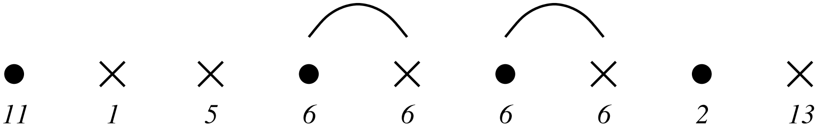

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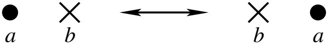
- 

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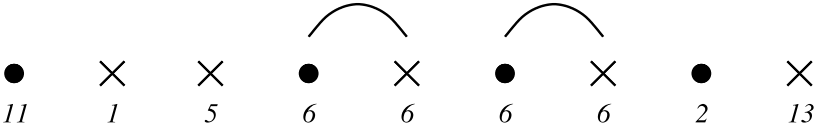

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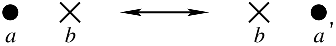
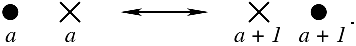
- 

- 

Combinatorial model

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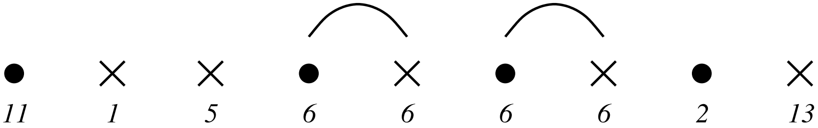

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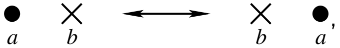
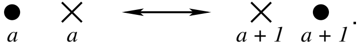
- 

- 

Remark: The size of S is independent of the base.

Combinatorial model

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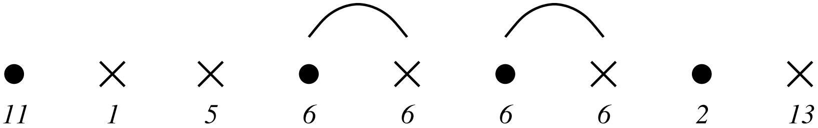

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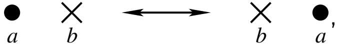
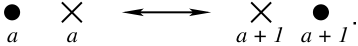
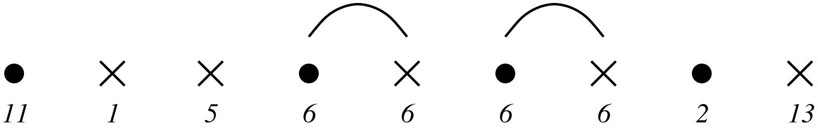
- 

- 

Condition ("dominance"):
In any base,

Combinatorial model

Arc diagram:



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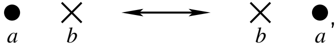
- 

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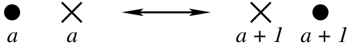
- 

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Combinatorial model

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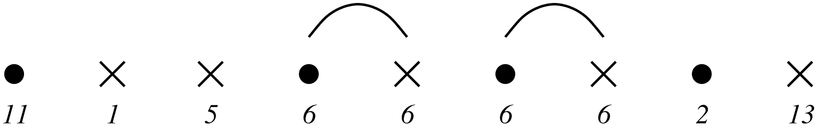

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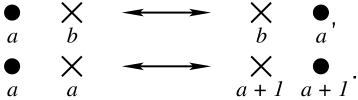

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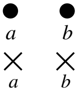

- $\Rightarrow a > b$,
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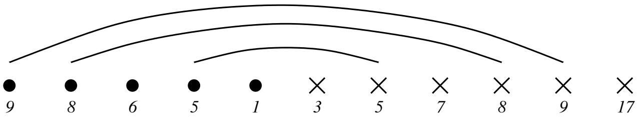

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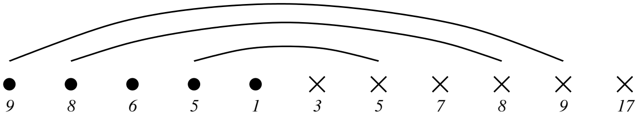

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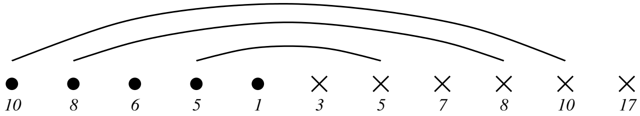

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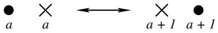

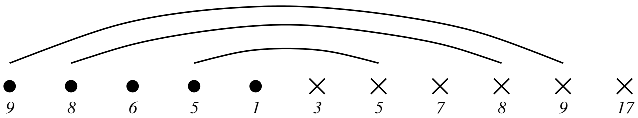

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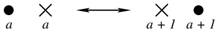

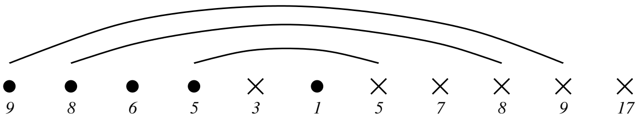

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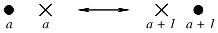

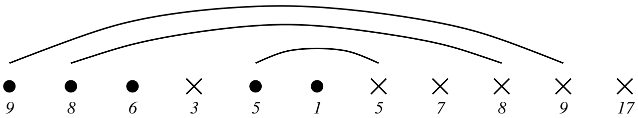

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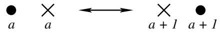

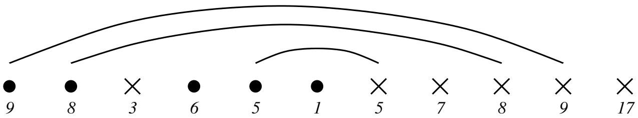

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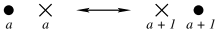

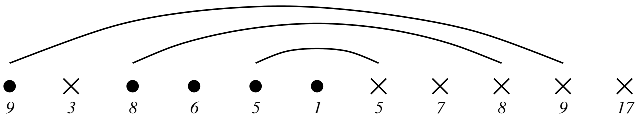

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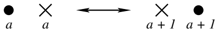

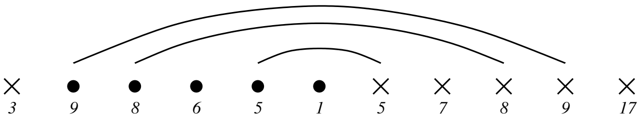

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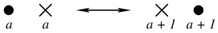

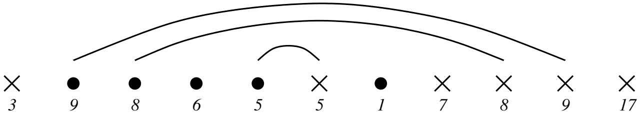

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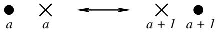

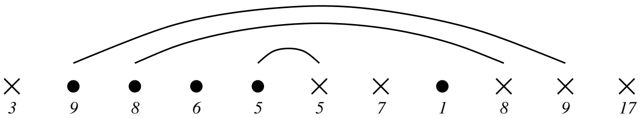

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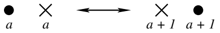

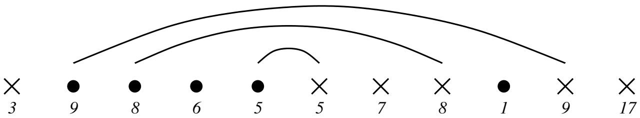

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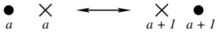

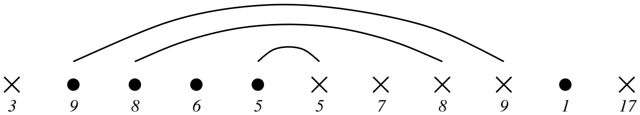

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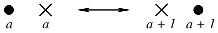

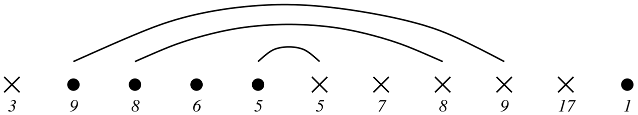

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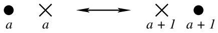

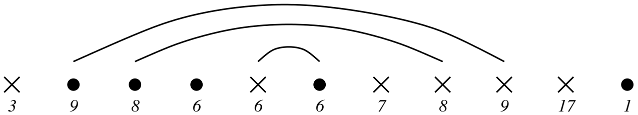

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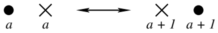

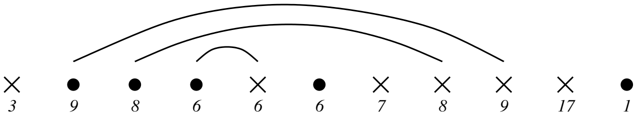

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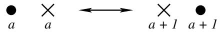

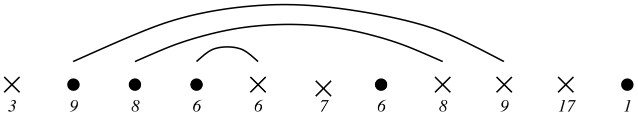

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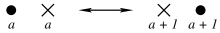

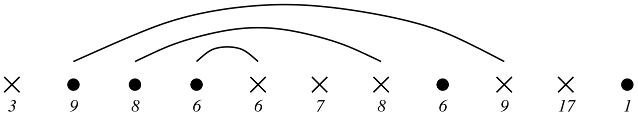

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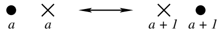

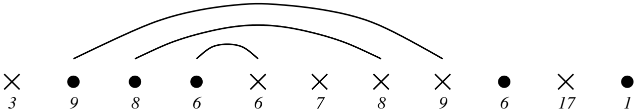

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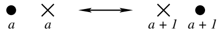

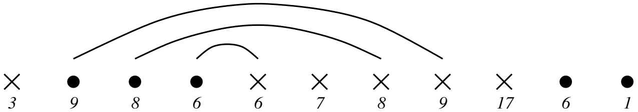

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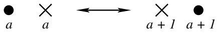

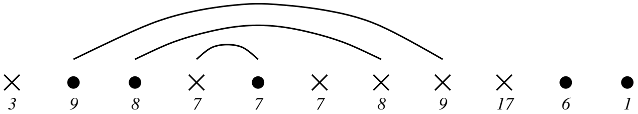

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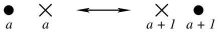

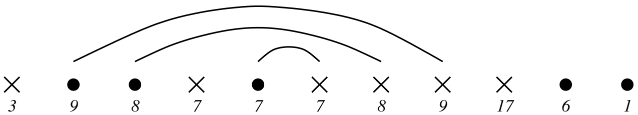

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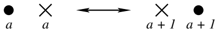

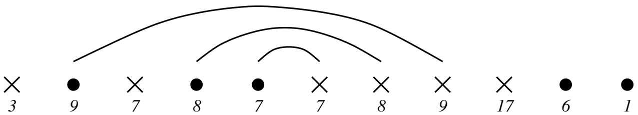

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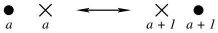

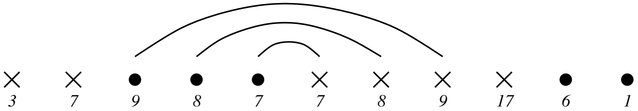

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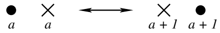

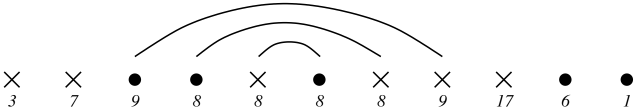

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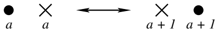

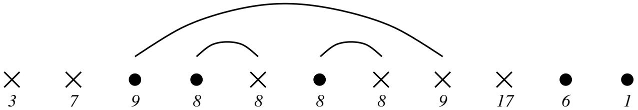

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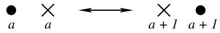

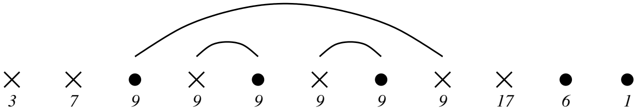

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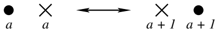

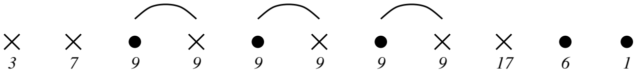

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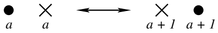

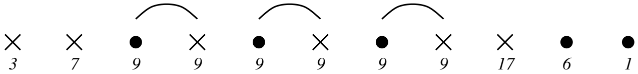

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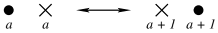

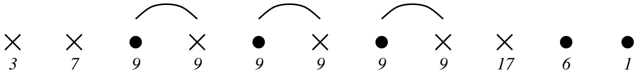

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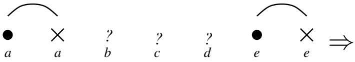

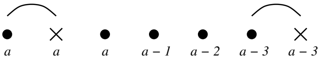

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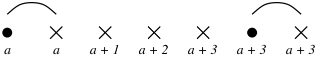

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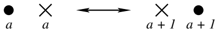

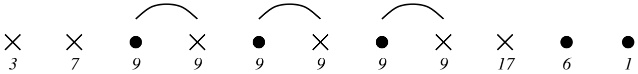

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Check that moves required to reach standard base preserve "interval property."

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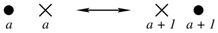

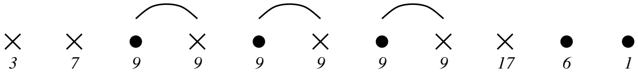

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Nest: Part of arc diagram under outermost arc.

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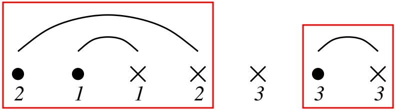

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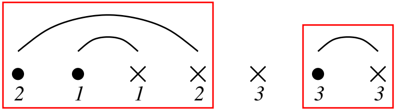

Proposition 2 (C., Hoyt, Reif)

If an arc diagram has at least two nests, then there exists a base where all arcs are short.

Example 3: Kac-Wakimoto character formula

Weyl character formula: For a semisimple Lie algebra,

$$ch\, L(\lambda) = \frac{\sum_{w \in W} (sgn\, w) \cdot w\left(e^{\lambda+\rho}\right)}{e^{\rho} R}.$$

Example 3: Kac-Wakimoto character formula

(Kac '77): If arc diagram has no arcs then

$$ch\, L(\lambda) = \frac{\sum_{w \in W} (sgn\, w) \cdot w\left(e^{\lambda+\rho}\right)}{e^{\rho} R}.$$

Example 3: Kac-Wakimoto character formula

(Bernstein, Leites '80): If arc diagram has one arc β then

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Example 3: Kac-Wakimoto character formula

(Kac-Wakimoto '94): Conjecturally, if arc diagram has only short arcs then

$$ch\, L(\lambda) = \frac{\sum_{w \in W} (sgn\, w) \cdot w \left(\frac{e^{\lambda+\rho}}{\prod_{\beta \in S}(1+e^{-\beta})} \right)}{(|S|)! e^{\rho} R}.$$

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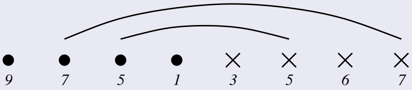
(C., Hoyt, Reif): See how SZ formula changes with each step of shortening algorithm. In the end, it is the Kac-Wakimoto formula. □

Example 4: Determinantal character formula

Theorem 2 (C., Hoyt, Reif)

For a totally connected arc diagram:



we have, with the convention $x_i = e^{\varepsilon_i}, y_j = e^{-\delta_j}$:

$$e^{\rho} R\, ch\, L(\lambda) = \pm \begin{vmatrix} x_1^9 & x_1^1 & f(x_1,y_1) & f(x_1,y_2) & f(x_1,y_3) & f(x_1,y_4) \\ x_2^9 & x_2^1 & f(x_2,y_1) & f(x_2,y_2) & f(x_2,y_3) & f(x_2,y_4) \\ x_3^9 & x_3^1 & f(x_3,y_1) & f(x_3,y_2) & f(x_3,y_3) & f(x_3,y_4) \\ x_4^9 & x_4^1 & f(x_4,y_1) & f(x_4,y_2) & f(x_4,y_3) & f(x_4,y_4) \\ 0 & 0 & y_1^3 & y_2^3 & y_3^3 & y_4^3 \\ 0 & 0 & y_1^6 & y_2^6 & y_3^6 & y_4^6 \end{vmatrix},$$

where $f(x_i, y_j) = \frac{(x_i y_j)^7}{1+(x_i y_j)^{-1}}$.

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- Classify all tame modules (ones for which the Kac-Wakimoto formula is correct in some base).
- Obtain a character formula for any base.
- Generalize to other Lie types.
- Connect the determinantal formula to supersymmetric functions.